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# MINIMUM-WEIGHT DESIGN OF BEAMS FOR MULTIPLE LOADING†

R. MAYEDA and W. PRAGER

Department of the Aerospace and Mechanical Engineering Sciences, University of California, San Diego
La Jolla, California

**Abstract**—The paper discusses minimum-weight design of beams of continuously varying cross section that have to carry any one of $n$ systems of loads. The weight per unit length is assumed to be given by $w(x) = a+bY(x)$, where $a$ and $b$ are constants and $Y(x)$ is the yield moment at the cross section $x$. A design method is presented that remains valid for the extreme cases $n = 1$ and $n = \infty$ and that enables the designer to prescribe a minimum value $Y_0$ below which the yield moment is not allowed to drop. The use of the method is illustrated by examples.

## 1. INTRODUCTION

THIS paper is concerned with minimum-weight design of beams that have to support any one of several systems of loads. Although beams of constant or piecewise constant cross section are preferable for practical reasons, beams of continuously varying cross section are discussed in the following, because they provide useful standards of comparison by which the efficiency of more practical designs may be judged.

Letting $x$ denote distance measured along the axis of the beam, we write the weight of the beam segment between the cross sections $x$ and $x+\mathrm{d}x$ as $w(x)\,\mathrm{d}x$ and assume that the unit weight $w(x)$ is given by

$$w(x) = a+bY(x), \tag{1.1}$$

where $a$ and $b$ are constants and $Y(x)$ is the yield moment at the cross section $x$. Within the limited range of cross sections suitable for a beam that has a given span and is to carry given loads, the actual relation between unit weight and yield moment can usually be linearized in this manner. For a rectangular sandwich section with a core of constant dimensions and identical thin face sheets of varying thickness, (1.1) represents the actual relation between $w$ and $Y$. Whenever the linear relation (1.1) is appropriate, minimizing the total weight of the beam means minimizing the integral of $Y(x)$ over the length of the beam.

Adopting the relation (1.1), Heyman [1] discussed minimum-weight design of beams for a single system of loads. On the other hand, Gross [2], Gross and Prager [3], Save and Prager [4], and Save and Shield [5] have treated minimum weight design of beams for moving loads, i.e. for infinitely many systems of loads. The methods used in [2] through [5] were specially devised for these problems rather than systematically developed from the method used in [1]. In this paper, minimum-weight design of beams is discussed for an arbitrary number $n$ of load systems. A method is presented that remains valid for the extreme cases $n = 1$ and $n = \infty$ and that enables the designer to prescribe a minimum value $Y_0$ below which the yield moment must not drop.

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## 2. GENERAL PRINCIPLES

Heyman [1] used a general theorem of Drucker and Shield [6], to show that a limit design for a given single system of loads is a minimum-weight design if there exists an associated yield mechanism with curvature rates of constant magnitude.† To discuss the restrictions that the condition $|v''(x)| = \text{const}$ imposes on the rate of deflection $v(x)$, consider a beam of the span $L$ that is built in at both ends [Fig. 1(a)]. If all loads acting on the beam have the same direction, statically admissible bending moments $M(x)$ (i.e. bending moments that are in equilibrium with these loads and satisfy the static boundary conditions) cannot vanish at more than two points. Indeed,

$$M(x) = M_0(x) + M(0)\left(1-\frac{x}{L}\right) + M(L)\frac{x}{L}, \tag{2.1}$$

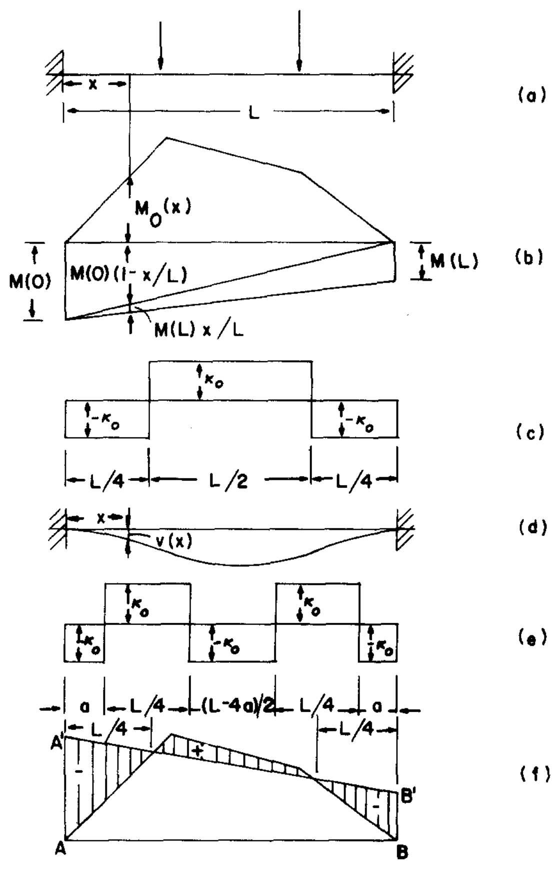


FIG. 1. (a) Built-in beam. (b) Statically admissible bending moments. (c) Rates of curvature of constant magnitude. (d) Yield mechanism of minimum-weight design. (e) Alternative rates of curvature of constant magnitude. (f) Bending moments of minimum-weight design.

† A "design" is specified by the function $Y(x)$. To verify that the design $Y(x)$ is a "limit design" for a given set of loads, one must find: (i) statically admissible bending moments $M(x)$ satisfying $|M(x)| \leq Y(x)$, and (ii) an "associated yield mechanism", i.e. continuously differentiable rates of deflection $v(x)$ satisfying the kinematic boundary conditions and furnishing rates of curvature $\varkappa(x) = -\mathrm{d}^2v(x)/\mathrm{d}x^2 = -v''(x)$ that are "compatible" with the bending moments $M(x)$. To be compatible with $M(x)$, the rate of curvature $\varkappa(x)$ must vanish where $|M(x)| < Y(x)$ and have the sign of $M(x)$ where $|M(x)| = Y(x)$.

where $M_0(x)$ is the statically determinate bending moment produced by the given loads in a simply supported beam of the same span, and $M(0)$ and $M(L)$ are the statically indeterminate clamping moments. The function $M_0(x)$ vanishes at the ends of the span and is convex if all loads have the same direction [Fig. 1(b)]. Accordingly, $M_0(x)$ has no zero for $0 < x < L$; we shall assume it to be positive in this open interval. The function $M(x)$ then vanishes at 0, 1, or 2 points in this interval according to whether 0, 1, or 2 of the values $M(0)$ and $M(L)$ are negative.

Wherever the rate of curvature of an associated yield mechanism does not vanish, its sign must be that of $M(x)$. If all loads have the same direction, the rate of curvature of any associated yield mechanism can thus have at most two sign changes in the span. This condition, together with the requirement that $|\varkappa| = |v''|$ should have a constant value, say $\varkappa_0$, uniquely determines the yield mechanism associated with the minimum weight design. Figure 1(c) shows the rate of curvature $\varkappa(x)$ and Fig. 1(d) the rate of deflection $v(x)$ of this mechanism. Note that the rate of curvature shown in Fig. 1(e) also has the constant magnitude $\varkappa_0$ and furnishes a rate of deflection that satisfies the kinematical boundary condition. This rate of deflection, however, does not represent a yield mechanism that is compatible with bending moments of the type shown in Fig. 1(b), because these bending moments have at most two sign changes in the span whereas the rate of curvature in Fig. 1(e) has four sign changes.

Because the rate of curvature of the yield mechanism associated with the minimum-weight design changes sign at the sections $x = L/4$ and $x = 3L/4$ [Fig. 1(c)], the bending moment $M(x)$ for the minimum-weight design must likewise change sign at these sections. This determines the "reactant line" A′B′ in Fig. 1(f) and hence the clamping moments AA′ and BB′, which are negative. The resulting bending moments $M(x)$ are indicated by shading in Fig. 1(f). The minimum-weight design of the beam, which is given by $Y(x) = |M(x)|$, may not be deemed acceptable because it has zero bending strength at $x = L/4$ and $x = 3L/4$. This can be avoided by prescribing a minimum value $Y_0$ below which the yield moment must not drop (see [7]).

Shield [8] has generalized the theorem of Drucker and Shield [6] to multiple loading of sandwich plates and shells, treating a simply supported circular plate under two alternative rotationally symmetric loadings as an example. Instead of deducing the criterion for minimum-weight design of beams under multiple loading from Shield's theorem, we will derive it from first principles, because this will be just as easy and, at the same time, enable us to work in the condition $Y(x) \geq Y_0$.

When a beam that is to carry any one of $n$ given systems of loads $S_1, S_2, \ldots, S_n$ has been designed for minimum-weight, the individual load system $S_i$ has turned out to be either critical or subcritical. In the first case, $S_i$ determines the variation of the yield moment $Y(x)$ of the minimum-weight design along some segments of the beam, the union of which will be denoted by $U_i$. If the load system $S_i$ is subcritical, it does not influence the minimum-weight design and may therefore be disregarded. It will be assumed that the load systems have been numbered in such a manner that only $S_1, S_2, \ldots, S_k$ with $k \leq n$ are critical.

The condition $Y(x) \geq Y_0$ may be critical or subcritical, that is, it may determine $Y(x)$ in some segments with the union $U_{k+1}$, or it may not be relevant for the minimum-weight design. Unless the contrary is stated explicitly, it will be assumed that the condition $Y(x) \geq Y_0$ is critical. The segments in $U_1, U_2, \ldots, U_{k+1}$ must then cover the entire span $U$ of the beam without overlap.

If $i \leq k$, the load system $S_i$ is critical and, hence represents a limit loading of the minimum-weight design $Y(x)$. Accordingly, there exist statically admissible bending moments $M_i(x)$ for this load system such that $|M_i(x)| \leq Y(x)$, with equality in $U_i$ only, and an associated yield mechanism $v_i(x)$ with curvature rates $\varkappa_i(x)$ that vanish identically except in $U_i$, where $\varkappa_i(x)$ and $M_i(x)$ have the same sign. If the power of the loads of $S_i$ on the rates of deflection $v_i(x)$ is denoted by $P_i$, we have

$$P_i = \int_{U_i} |\varkappa_i(x)|\, Y(x)\, \mathrm{d}x. \tag{2.2}$$

Next, let $Y^*(x) \geq Y_0$ be a "safe design" for the load systems $S_1, S_2, \ldots, S_k$, that is, a design for which none of these load systems represents a limit loading. Testing this design with the yield mechanism $v_i(x)$, we find

$$P_i < \int_{U_i} |\varkappa_i(x)|\, Y^*(x)\, \mathrm{d}x. \tag{2.3}$$

It follows from (2.2) and (2.3) that

$$\int_{U_i} |\varkappa_i(x)|\, Y(x)\, \mathrm{d}x < \int_{U_i} |\varkappa_i(x)|\, Y^*(x)\, \mathrm{d}x. \tag{2.4}$$

If the curvature rates $\varkappa_i(x)$ satisfy

$$\left.\begin{array}{ll} \varkappa_i(x) = 0 & \text{if } x \notin U_i \\ |\varkappa_i(x)| = \varkappa_0 & \text{if } x \in U_i \end{array}\right\} \text{for } i = 1, 2, \ldots, k, \tag{2.5}$$

where $\varkappa_0$ is a constant, addition of the relations (2.4) for $i = 1, 2, \ldots, k$ furnishes

$$\int_{U - U_{k+1}} Y(x)\, \mathrm{d}x < \int_{U - U_{k+1}} Y^*(x)\, \mathrm{d}x. \tag{2.6}$$

Now,

$$\int_{U_{k+1}} Y(x) \leq \int_{U_{k+1}} Y^*(x)\, \mathrm{d}x, \tag{2.7}$$

because $Y(x) = Y_0$ and $Y^*(x) \geq Y_0$ in $U_{k+1}$. From (2.6) and (2.7) there follows the inequality

$$\int_U Y(x)\, \mathrm{d}x < \int_U Y^*(x)\, \mathrm{d}x. \tag{2.8}$$

In view of (1.1), this inequality shows that the design $Y(x)$ has a smaller weight than the safe design $Y^*(x)$ no matter how close the latter may come to being a limit design. Note that (2.6) directly establishes the superiority of the design $Y(x)$ if the condition $Y(x) \geqq Y_0$ is subcritical.

## 3. EXAMPLES

(a) The first example concerns the minimum-weight design of a beam of the span $L = 4l$ that is built in at both ends and subjected to alternate load systems $S_1$ and $S_2$. The system $S_1$ consists of a total load $8P_1$ that is uniformly distributed over the span

[Fig. 2(a)]. The system $S_2$ consists of a concentrated load $2P_2$ that acts at the center of the span [Fig. 2(b)]. The yield moment $Y(x)$ of the desired minimum-weight design is nowhere to drop below the given value $Y_0$.

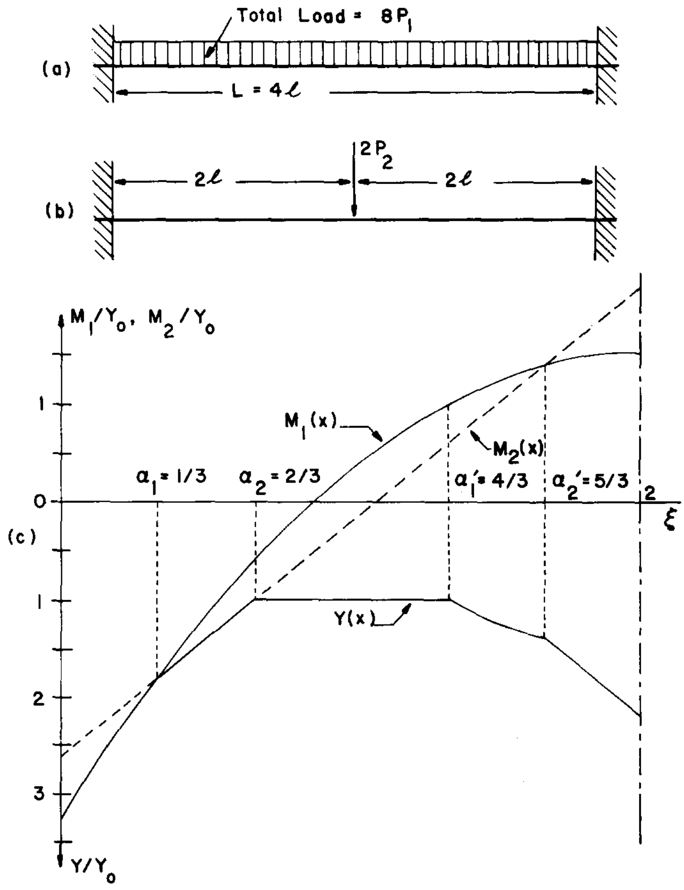


FIG. 2. (a), (b) Load systems. (c) Bending moments and yield moment.

On account of the symmetry of load and support, we need only be concerned with the variation of $Y(x)$ over half the span. Assuming first that both load systems and the lower bound on $Y(x)$ are critical, we tentatively divide the semi-span into the five intervals listed in Table 1. For the load systems $S_1$ and $S_2$, we must then find yield mechanisms with the following rates of curvature

$$\kappa_1(x) = \begin{cases} -\kappa_0 \text{ in } I_1, \\ \kappa_0 \text{ in } I_4, \\ 0 \text{ elsewhere;} \end{cases} \tag{3.1}$$

$$\kappa_2(x) = \begin{cases} -\kappa_0 \text{ in } I_2, \\ \kappa_0 \text{ in } I_5, \\ 0 \text{ elsewhere.} \end{cases} \tag{3.2}$$

TABLE 1

| Interval | | Design governed by |
|---|---|---|
| Designation | Definition | |
| $I_1$ | $0 \leq x < a_1$ | $S_1$ |
| $I_2$ | $a_1 < x < a_2$ | $S_2$ |
| $I_3$ | $a_2 < x < a_1'$ | $Y(x) \geq Y_0$ |
| $I_4$ | $a_1' < x < a_2'$ | $S_1$ |
| $I_5$ | $a_2' < x \leq 2l$ | $S_2$ |

Because the rates of deflection $v_1(x)$ associated with the curvature rates (3.1) must have vanishing first derivatives at $x = 0$ and $x = 2l$, the intervals $I_1$ and $I_4$ must have the same length, and a similar remark applies to the intervals $I_2$ and $I_5$. Accordingly,

$$a_1' = 2l - a_2, \qquad a_2' = 2l + a_1 - a_2. \tag{3.3}$$

In the following, the notations

$$\xi = x/l, \qquad \alpha_1 = a_1/l, \qquad \alpha_2 = a_2/l \tag{3.4}$$

will be used. The load systems $S_1$ and $S_2$ produce bending moments of the form

$$M_1(\xi) = P_1 l\xi(4-\xi) - C_1, \tag{3.5}$$

$$M_2(\xi) = P_2 l\xi - C_2, \tag{3.6}$$

where $C_1$ and $C_2$ are clamping moments. The values of $\alpha_1$, $\alpha_2$, $C_1$, and $C_2$ must be obtained from the following conditions [see Fig. 2(c), in which $\alpha_1 = \frac{1}{3}$, $\alpha_2 = \frac{2}{3}$, while the full-line curve represents the bending moments $M_1(x)$ and the dashed straight line the bending moments $M_2(x)$]

$$\left.\begin{aligned} &M_1(\alpha_1) = M_2(\alpha_1), \\ &M_1(\alpha_2') = M_2(\alpha_2'), \qquad \alpha_2' = 2+\alpha_1-\alpha_2, \\ &M_2(\alpha_2) = -Y_0, \\ &M_1(\alpha_1') = Y_0, \qquad \alpha_1' = 2-\alpha_2. \end{aligned}\right\} \tag{3.7}$$

These equations are linear in the dimensionless quantities

$$P_1' = P_1 l/Y_0, \qquad P_2' = P_2 l/Y_0, \qquad C_1' = C_1/Y_0, \qquad C_2' = C_2/Y_0, \tag{3.8}$$

but nonlinear in $\alpha_1$ and $\alpha_2$. It is therefore expedient to express the quantities (3.8) as functions of the parameters $\alpha_1$ and $\alpha_2$. With

$$\Delta = 4 - 2\alpha_1 - 2\alpha_2 - \alpha_1^2 + 3\alpha_1\alpha_2 - 2\alpha_2^2, \tag{3.9}$$

we find

$$\left.\begin{aligned} P_1' &= 2/\Delta, \qquad P_2' = 2(2-2\alpha_1+\alpha_2)/\Delta, \\ C_1' &= \frac{2}{\Delta}(4-\alpha_2^2)-1, \\ C_2' &= \frac{2}{\Delta}(2-2\alpha_1+\alpha_2)\alpha_2+1. \end{aligned}\right\} \tag{3.10}$$

The relations (3.10) are only valid when the load systems $S_1$ and $S_2$ as well as the lower bound for the yield moment are critical. Limits of validity of these relations are therefore reached when the length of one of the intervals $I_1$, $I_2$, or $I_3$ becomes zero. This yields the conditions $\alpha_1 = 0$, $\alpha_1 = \alpha_2$, or $\alpha_2 = \alpha_1'$, i.e. $\alpha_2 = 1$. We now propose to rewrite these conditions in terms of $P_1'$ and $P_2'$. According to (3.10), we have

$$R = P_2'/P_1' = 2-2\alpha_1+\alpha_2. \tag{3.11}$$

Considering first the case $\alpha_1 = 0$, we have

$$R = 2+\alpha_2 \tag{3.12}$$

and hence $2 \leq R \leq 3$ since $0 \leq \alpha_2 \leq 1$. With $\alpha_1 = 0$ and (3.12), it follows from (3.9) and (3.10) that

$$\Delta = 2R(3-R), \qquad P_1' = 1/(3R-R^2). \tag{3.13}$$

Multiplying both sides of the second equation (3.13) by $(3R-R^2)P_1'$, using $P_1'R = P_2'$, and solving for $P_1'$, we obtain

$$P_1' = P_2'^2/(3P_2'^2-1) \tag{3.14}$$

as the value of $P_1'$ at which the load system $S_1$ becomes subcritical. Whenever $P_1'$ is smaller than the value in (3.14), the system $S_1$ is therefore subcritical.

Next, we discuss the case $\alpha_1 = \alpha_2$. Here,

$$R = 2-\alpha_1 \tag{3.15}$$

and hence $1 \leq R \leq 2$. We have

$$\Delta = 4(R-1), \qquad P_1' = 1/[2(R-1)], \tag{3.16}$$

or, multiplying the last equation by $2(R-1)$ and using $P_1'R = P_2'$,

$$P_2' = P_1'+\tfrac{1}{2}. \tag{3.17}$$

Since this is the value of $P_2'$ at which the system $S_2$ becomes subcritical, this system is subcritical whenever $P_2'$ is smaller than the value in (3.17).

Finally, for $\alpha_2 = 1$, we have

$$R = 3-2\alpha_1 \tag{3.18}$$

and hence $1 \leq R \leq 3$. Accordingly,

$$\Delta = \tfrac{1}{4}(3-R)(R-1), \qquad P_1' = 8/(4R-R^2-3). \tag{3.19}$$

From the second of these equations and the definition (3.8) of $P'_1$, we find

$$Y_0 = \tfrac{1}{8}P'_1(4R-R^2-3) \tag{3.20}$$

as the value of $Y_0$ at which the condition $Y(x) \le Y_0$ becomes subcritical. This condition thus is subcritical whenever $Y_0$ is smaller than the value in (3.20).

Returning now to the case where both load systems as well as the lower bound on $Y(x)$ are critical, we use (3.11) to express $\alpha_2$ as a linear function of $\alpha_1$ and $R$. Substituting this into (3.9) and the first equation (3.10), we obtain a quadratic equation for $\alpha_1$ with the relevant root

$$\alpha_1 = \tfrac{1}{6}\{4-5R+\sqrt{[R^2+32R+16-(24/P'_1)]}\}. \tag{3.21}$$

When the value of $\alpha_1$ is known, that of $\alpha_2$ is obtained from (3.11):

$$\alpha_2 = R+2\alpha_1-2. \tag{3.22}$$

Figure 2(c) corresponds to $P'_1 = 1{\cdot}2$, $P'_2 = 2{\cdot}4$, and hence $R = 2$, $\alpha_1 = \frac{1}{3}$, $\alpha_2 = \frac{2}{3}$.

(b) The second example concerns the minimum-weight design of a beam of the span $L$ that is simply supported at the end $x = 0$, built in at the end $x = L$, and subjected to three alternative load systems (Figs. 3a, b, c). The load system $S_k (k = 1, 2, 3)$ consists of a single load of the intensity $P$ acting at the cross section $x = kL/5$. To facilitate comparison of the present minimum-weight design with that obtained by Gross and Prager [2, 3] for a

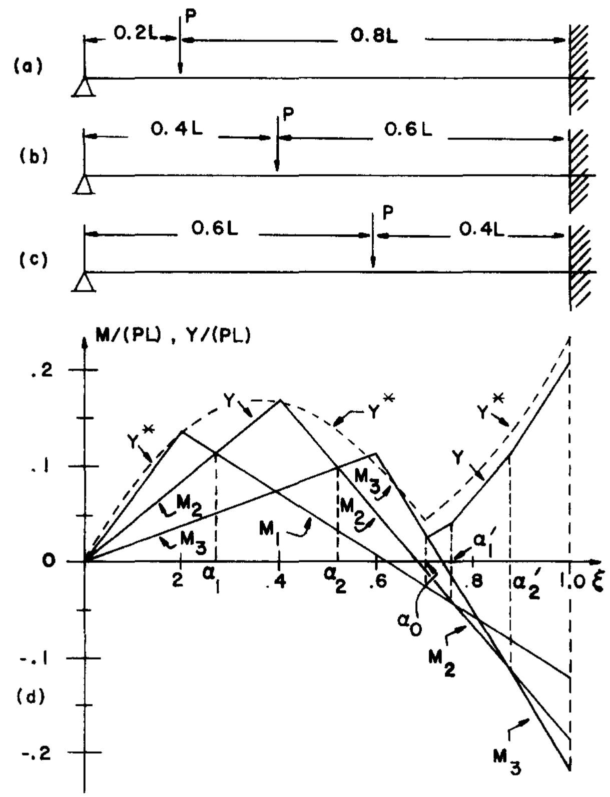


FIG. 3. (a), (b), (c) Load systems. (d) Bending moments and yield moment.

load of fixed intensity and arbitrary point of application ("moving load"), we refrain from imposing a lower bound on the yield moment $Y_0(x)$.

Assuming that all three load systems are critical, we tentatively divide the span into the six intervals listed in Table 2. In the following, the notations

$$\left.\begin{aligned} \alpha_k &= a_k/L, \quad \alpha'_k = a'_k/L, \quad (k = 1, 2, 3) \\ \alpha_0 &= a_0/L, \quad \xi = x/L \end{aligned}\right\} \tag{3.23}$$

will be used.

TABLE 2

| Interval | | Design governed by |
|---|---|---|
| Designation | Definition | |
| $I_1$ | $0 \leq x < a_1$ | $S_1$ |
| $I_2$ | $a_1 < x < a_2$ | $S_2$ |
| $I_3$ | $a_2 < x < a_0$ | $S_3$ |
| $I_4$ | $a_0 < x < a'_1$ | $S_1$ |
| $I_5$ | $a'_1 < x < a'_2$ | $S_2$ |
| $I_6$ | $a'_2 < x < L$ | $S_3$ |

For the load system $S_1$, we must find a yield mechanism $v_1(x)$ that satisfies the kinematic boundary conditions $v_1(0) = v_1(L) = v'_1(L) = 0$ and has the rates of curvature

$$\varkappa_1(x) = \begin{cases} \varkappa_0 \text{ in } I_1, \\ -\varkappa_0 \text{ in } I_4, \\ 0 \text{ elsewhere.} \end{cases} \tag{3.24}$$

Since $\varkappa_1(x) = -v''_1(x)$, the kinetic boundary conditions furnish the relation

$$\alpha'^2_1 = \alpha^2_1 + \alpha^2_0. \tag{3.25}$$

Similarly, the yield mechanism for $S_2$ furnishes $\alpha'^2_2 = \alpha^2_2 - \alpha^2_1 + \alpha'^2_1$ or, on account of (3.25),

$$\alpha'^2_2 = \alpha^2_2 + \alpha^2_0. \tag{3.26}$$

Finally, from the yield mechanism for $S_3$, we obtain $1 = \alpha^2_0 - \alpha^2_2 + \alpha'^2_2$ or, on account of (3.26),

$$\alpha_0 = \tfrac{1}{2}\sqrt{2}. \tag{3.27}$$

The bending moment associated with the load system $S_1$ is given by

$$M_1(\xi) = \begin{cases} R_1 L\xi, & 0 \leq \xi \leq 0{\cdot}2, \\ R_1 L(\xi - 0{\cdot}2) & 0{\cdot}2 \leq \xi \leq 1, \end{cases} \tag{3.28}$$

where $R_1$ is the unknown reaction at the simply supported end. Similar expressions can be written for $M_2(\xi)$ and $M_3(\xi)$. The values of $\alpha_1$, $\alpha_2$, $R_1$, $R_2$, and $R_3$ must be obtained from the conditions

$$\left.\begin{aligned} M_1(\alpha_1) &= M_2(\alpha_1), & M_1(\alpha_1') &= M_2(\alpha_1'), \\ M_2(\alpha_2) &= M_3(\alpha_2), & M_2(\alpha_2') &= M_3(\alpha_2'), \\ M_1(\alpha_0) &= -M_3(\alpha_0). & & \end{aligned}\right\} \tag{3.29}$$

Eliminating the reactions from first four of these conditions, and using (3.25) through (3.27), one obtains biquadratic equations for $\alpha_1$ and $\alpha_2$, which yield

$$\alpha_1 = 0{\cdot}2718, \qquad \alpha_2 = 0{\cdot}5183 \tag{3.30}$$

and hence

$$\alpha_1' = 0{\cdot}7571, \qquad \alpha_2' = 0{\cdot}8765. \tag{3.31}$$

With the values (3.30) and (3.31), the reactions are readily obtained from (3.29). One finds

$$R_1 = 0{\cdot}6802P, \qquad R_2 = 0{\cdot}4164P, \qquad R_3 = 0{\cdot}1884P.$$

Figure 3(d) shows the bending moments $M_1$, $M_2$, $M_3$, and the yield moment $Y$ of the minimum-weight design. The dotted curve indicates the yield moment $Y^*$ of the minimum-weight design for a moving load (see [2–4]). At the points of application of the alternative loads considered here, the values of $Y$ very slightly exceed those of $Y^*$; for $\xi \geq \alpha_0$, however, $Y^*$ consistently exceeds $Y$.

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**Résumé**—L'exposé examine le dimensionnement à poids minimum de poutres à section variant d'une façon continue qui ont à supporter l'un quelconque de $n$ systèmes de charges. Le poids par unité de longueur est supposé être donné par $w(x) = a+bY(x)$, où $a$ et $b$ sont des constantes et $Y(x)$ est le moment de limite élastique à la section $x$. Une méthode de dimensionnement est présentée qui reste valable pour les cas extrêmes où $n = 1$ et $n = \infty$ et ceci permet au dessinateur d'établir une valeur minimum $Y_0$ au-dessous de laquelle il n'est pas permis au moment de limite élastique de baisser. L'emploi de la méthode est illustré par des exemples.

**Zusammenfassung**—Diese Arbeit behandelt Minimalgewichts-Entwürfe für Träger mit stetig veränderlichen Querschnitten, die jedes von $n$ Lastsystemen tragen können. Das Gewicht je Längeneinheit wird als $w(x) = a+bY(x)$

angenommen, wobei $a$ und $b$ Kostanten sind, und $Y(x)$ das Fliessmoment beim Querschnitt $x$. Eine Entwurfsmethode wird gegeben, die bei Extremfällen $n = 1$ und $n = \infty$ gültig bleibt und die es erlaubt den Minimalwert $Y_0$ vorzuschreiben, unter welchen das Fliessmoment nicht absinken darf. Die Anwendung der Methode wird gezeigt.

**Абстракт**—В работе рассуждается вопрос проектирования на минимум веса балок, с изменяющимся непрерывно поперечным сечением, которые должны воспринимать некоторое из $n$ систeмов нагрузки. Вес на единицу длины задан в форме $w(x) = a+bY(x)$, где $a$ и $b$ являются константами, $aY(x)$ обозначает момент течения поперечного сечения $x$. Метод проектирования таков, что заключает значение для экстремальных случаев, т.е. для $n = 1$ и $n = \infty$. Позваляет также проектировщику определить минимальное значение $Y_0$, ниже которого не допускается снижать момент течения. Примеры иллюструют использование метода.